Fuzzy Set Theoretical Approach to the Tone Triangular System

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***Abstract*—The present study considers a fuzzy color system in which three input fuzzy sets are constructed on the tone triangle. This system can process a fuzzy input to a tone triangular system and output to a color on the RGB triangular system. Three input fuzzy sets (*not black*, *white*, and *light*) are applied to the tone triangle relationship. By treating three attributes of chromaticness, whiteness, and blackness on the tone triangle, a target color can be easily obtained as the center of gravity of the output fuzzy set. In the present paper, the differences between fuzzy inputs and inference outputs are described, and the relationship between inference outputs for crisp inputs and for fuzzy inputs on the RGB triangular system are shown by the input-output characteristics between chromaticness, whiteness, and blackness as the inputs and redness (as one of the outputs).**

***Index Terms*—fuzzy set theoretical approach, tone triangle, color triangle, RGB triangle, additive color mixture, fuzzy set of triangular pyramid, conical fuzzy input, tone-hue mapping, fuzzy rules, vague color, equilateral triangle**

I. Introduction

Using the additive color mixing reported in recent studies [6], [7], the relationship between input fuzzy sets on the RGB triangle and fuzzy inputs of conical membership functions was examined. The color triangle (plane) represents the hue and saturation of a color. The six fundamental colors and white can be represented on the same color triangle (See Fig. 3*b*). Vague colors on the color triangle and chromaticity diagram were clarified.

However, a technique for obtaining expressions of the tone triangle in the RGB system using the fuzzy set theoretical method and the additive color mixture has not been reported. In the present study, the relationship between two or three input fuzzy sets on the tone triangle (See Fig. 3*a*) and fuzzy inputs of conical membership functions is examined. The six fundamental colors and white can be represented on the RGB triangle (which is similar to the color triangle in Fig. 3*b*). Vague colors on the tone triangle are clarified. Such a system will help us to determine the average color value as the center of gravity of the attribute information of vague colors. This fuzzy set theoretical approach is useful for vague color information processing, color identification, and similar applications.

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II. Methods

A. Color Triangle and Additive Color Mixture

Additive color mixing occurs when two or three beams of differently colored light combine. It has been found that mixing just three additive primary colors, red, green, and blue, can produce the majority of colors. In general, a color vector can be described by certain quantities as a scalar and a direction. These quantities are referred to as the tri-stimulus values, *R* for the red component, *G* for the green component, and *B* for the blue component, and are given as follows:

$$\vec{C} = \vec{R} + \vec{G} + \vec{B} \quad (1)$$

This is referred to as the RGB color model (Fig. 1), which allows colors to be represented by a planar diagram. The RGB color model can be used to draw the red, green, and blue components (*R*, *G*, *B*) on the axis of color space (including to a color triangle), as shown in Fig. 1. The coordinates (*r*, *g*, *b*) on the color triangle can specify various colors. The coordinates correspond to the amounts of *R*, *G*, and *B* that make up the color. The coordinates specifying the center of the color triangle represent the case in which the three primary colors are mixed in equal proportion and indicate the color white. Such representations are referred to as chromaticity diagrams. A chromaticity diagram represents hue and saturation, but not lightness [12]. On the color triangle (the dotted area in Fig. 1) [13], the percentages of redness, greenness, and blueness, where the total of the three attributes is equivalent to 100%, specify a color. In order to indicate only the direction of a color vector, i.e., the chromaticity, the redness *r*, greenness *g*, and blueness *b* are obtained as follows:

$$r = \frac{R}{R+G+B} \quad (2)$$

$$g = \frac{G}{R+G+B} \quad (3)$$

$$b = \frac{B}{R+G+B} \quad (4)$$

$$r + g + b = 1 \quad (5)$$

In other words, the direction is shown as the ratio of tri-stimulus values *R*, *G*, and *B*. The total of these ratios is equal to 1, as shown in Eq. (5).

In Fig. 1, at red *R*, the components are (*R*, *G*, *B*) = (1, 0, 0) and the coordinates are (*r*, *g*, *b*) = (1, 0, 0). At red, green, blue, the components are (*R*, *G*, *B*) = (*r*, *g*, *b*). At yellow, for instance, the components are (*R*, *G*, *B*) = (1, 1, 0) and the coordinates are (*r*, *g*, *b*) = (0.5, 0.5, 0). Colors on three squares: WMRY, WYGCy, and WCyBM in Fig. 2*b* (color space) corresponds to those on three diamonds: WMRY, WYGCy, and WCyBM in Fig. 3*b* (color triangle). Thus, most of the colors on part of the surface of a color space can be displayed in the color triangle. See Appendix C (Table 1).

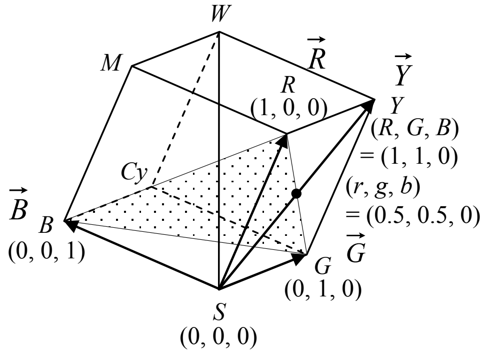


Figure 1. A color triangle and additive color mixture. The origin is black *S*.

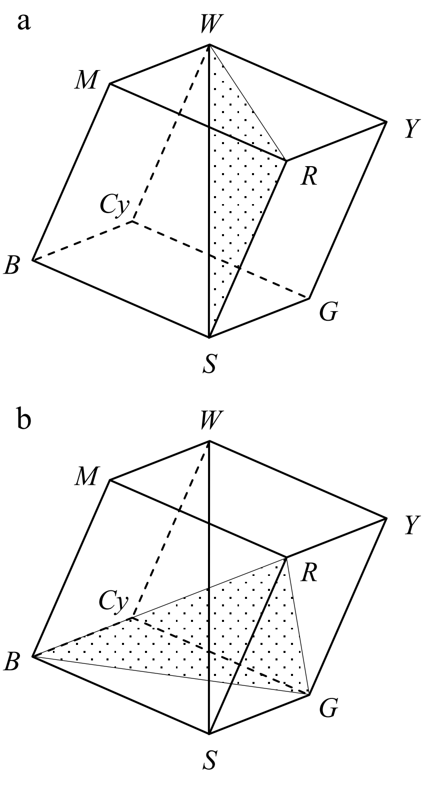


Figure 2. (*a*) A tone triangle and (*b*) a color triangle in the same color space. *S* is black. *Cy* is cyan.

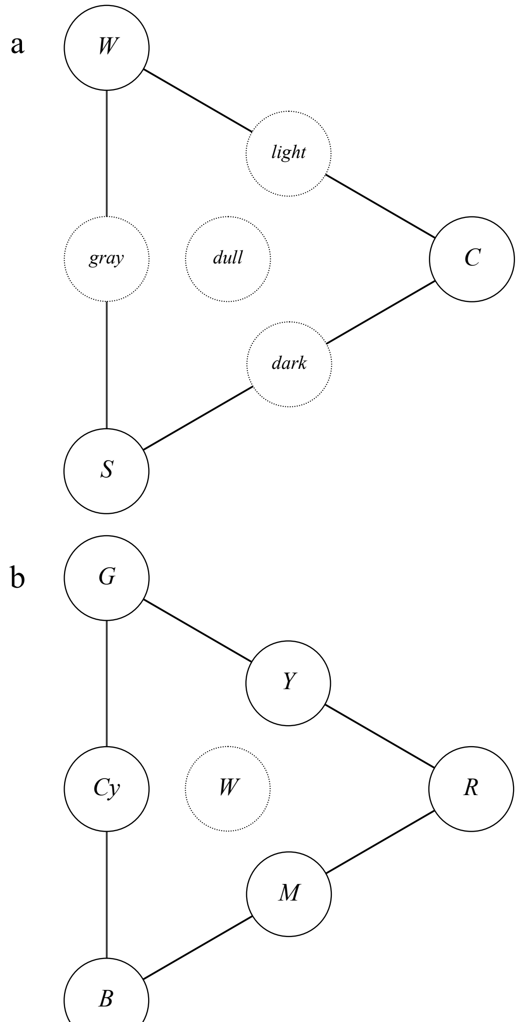


Figure 3. (*a*) A tone triangle. A point in the plane of the triangular system represents the lightness and saturation of a color. *S* is black. *C* is the maximum chromaticness of each hue. (*b*) A color triangle. A point in the plane of the triangular system represents the hue and saturation of a color. *Cy* is cyan.

B. Tone Triangle and Color Triangle Designs

In RGB color space (Fig. 2), the tone triangle and the color triangle are considered. In the color space, for instance, when the hue of a color is red *R*, the tone triangle and the color triangle are fixed as shown in Fig. 2*a* and *b*. The intersection between the tone triangle and the color triangle corresponds to line *W*-*R* in Fig. 3*b* (not in Fig. 2). That is, a coordinate on the tone triangle (CWS) is mapped to the line *W*-*R* on the RGB triangle. The color triangle (which covers the colors on part of the surface of the color space) and the RGB triangle (which covers all of the colors of the color space) designed as the output in this fuzzy system are obviously different. In Fig. 3*a*, *C* is the maximum chromaticness (No.66: red *R* in Fig. 4*a*) as a hue, but *Cy* in Fig. 3*b* is cyan, midway between blue and green (No.6: cyan *Cy* in F ig. 4*b*).

The color triangle in Fig. 2*b* is an equilateral triangle, if the hue is red *R*, the tone triangle is a right triangle. When the hue of a color is orange, i.e., midway between yellow and red, the tone triangle is nearly an isosceles triangle.

By shaping another triangle into an equilateral triangle, it is possible to normalize the equilateral coordinates in the fuzzy system.

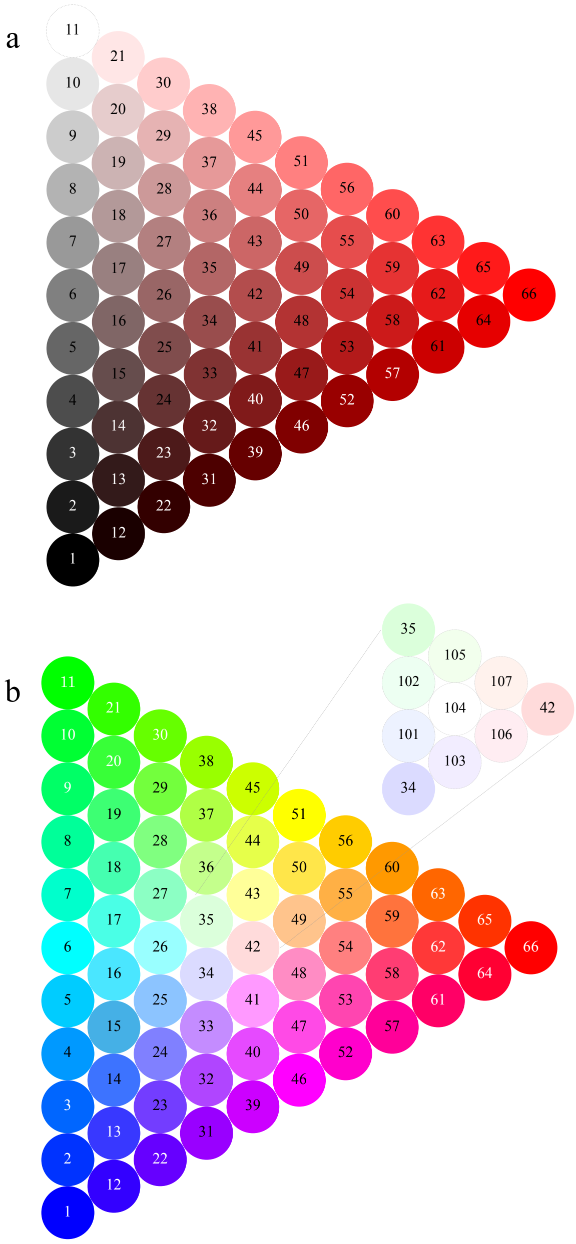


Figure 4. (*a*) Sixty-six crisp color inputs on the tone triangle. (*b*) Sixty-six crisp color inputs and white with six neighboring colors (*detail*) on the color triangle.

The present study considers a system of the three primary colors, red, green, and blue (RGB), presented on an color triangle. As Fig. 3*b* shows, blue, cyan, green, yellow, red, magenta, and white are abbreviated as *B*, *Cy*, *G*, *Y*, *R*, *M*, and *W*, respectively. Six fundamental color coordinates, e.g., (r_1, g_1, b_1), (r_6, g_6, b_6), (r_{11} , g_{11}, b_{11}), …, were selected, where r_n, g_n, and b_n are the redness, greenness, and blueness attributes, respectively, of the n^{th} color. When the hue of a color (e.g., red, green, or blue) is fixed on the color triangle in Fig. 3*b*, the color exists on the tone triangle in Fig. 3*a*. As Fig. 3*a* shows, maximum chromaticness, white, and black are abbreviated as *C*, *W*, and *S*, respectively. Dark (or deep), light (or pale), and dull are modifiers.

The tone triangle and color triangle in Fig. 4 correspond to the schematic diagrams shown in Fig. 3. The color names or modifiers in Fig. 4*a* are No.1: black, No.6: gray, No.11: white, No.46: dark (or deep), No.51: light (or pale), and No.66: *C*, maximum chromaticness [3] (e.g., vivid red). The color names in Fig. 4*b* are No.1: blue, No.6: cyan, No.11: green, No.51: yellow, No.66: red, No.46: magenta, and No.104: white.

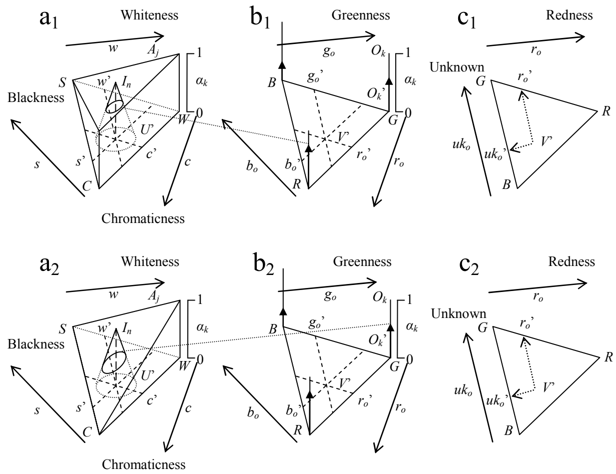


Figure 5. Fuzzy system using the membership function of input fuzzy sets A_j with (*a*) conical fuzzy input I_n on the tone triangle, (*b*) output crisp sets O_k on the RGB triangle, and (*c*) a color coordinate on the graphic plane in each trace. a_1: input fuzzy set A_1 (*not black*) and a_2: input fuzzy set A_0 (*white*).

C. Fuzzy Rules

Figure 5 illustrates a fuzzy system consisting of input fuzzy sets, fuzzy input on the tone triangle, crisp output, fuzzy output on the RGB triangle, and crisp output on the graphic plane. Figure 5*a* shows that the directions of the three center lines of the tone triangle (CWS) are chromaticness, whiteness, and blackness. With the increasing blackness, the membership value μ_j (black component) of the input fuzzy set declines in a_1. This forms an input fuzzy set of *not black* (fuzzy complement of *black*). On the other hand, with increasing whiteness, the membership value μ_j (white component) of the input fuzzy set increases in a_2. This forms an input fuzzy set of *white*.

In this fuzzy system, A_j is a fuzzy set of inputs, I_n is a fuzzy set of fuzzy inputs, O_k is a crisp set of outputs, and O_k' is a fuzzy set of outputs. In this case, Fig. 5*a* shows the triangular coordinates (*c*, *w*, *s*) on the tone triangle, and the relationship between chromaticness *c*, whiteness *w*, and blackness *s*, is as follows:

$$c + w + s = 1 \quad (6)$$

Figure 5*b* shows the triangular coordinates (r_o, g_o, b_o) on the RGB triangle (similar to the color triangle). Figure 5*c* shows the general coordinates (r_o', uk_o') on the graphic plane. Using two input fuzzy sets of different triangular pyramids in a_1 and a_2, the fundamental colors (or the fundamental colors with vagueness (i.e., orange for red)) can be processed (as a fuzzy inference). In this case, chromaticness is treated as the saturation.

The fuzzy rules are as follows (See Figs. 5 and 8):

$$R^k : if\ U\ is\ A_j\ then\ V\ is\ O_k \qquad (7)$$

where k is the rule number (k = 1, 2, 3), corresponding to R, G, and B components. Here, A_j is a fuzzy set of inputs (antecedent) (j = 1, 2), and O_k is a crisp set of outputs (consequent), where k does not correspond to j (Table 1). In addition, $U = (c, w, s)$ is the input coordinate, and $V = (r_o, g_o, b_o)$ is the output coordinate.

TABLE I.
FUZZY RULES FOR SIX FUNDAMENTAL COLORS

Hue color	Input fuzzy set			Output crisp set		
	R^1	R^2	R^3	R^1	R^2	R^3
Red	A_1	A_0	A_0	O_1	O_2	O_3
Green	A_0	A_1	A_0	O_1	O_2	O_3
Blue	A_0	A_0	A_1	O_1	O_2	O_3
Yellow	A_1	A_1	A_0	O_1	O_2	O_3
Cyan	A_0	A_1	A_1	O_1	O_2	O_3
Magenta	A_1	A_0	A_1	O_1	O_2	O_3

Membership values of A_1 and A_0 are 1.0 and 0.0 at coordinates (100, 0, 0) in %.

Here, U is fixed on the tone triangle, and V is fixed on the RGB triangle. A fuzzy set A_j of inputs shows a triangular pyramid shape with corners C, W, and S on the tone triangle (Fig. 5*a*), and a crisp set O_k of outputs for rule R^k is shown as a singleton (vertical pole) at corner R, G, or B (fuzzy set O_k' indicated by vertical arrows in Fig. 5*b*) on the RGB triangle. The output is O_k if the input is A_j. The relationship between A_j and O_k is shown in Table 1.

Table 1 shows the fuzzy rules of six fundamental colors. Three rules (R^1, R^2, R^3) are composed of two input fuzzy sets (Figs. 5 and 6). Red components (R, G, B) = (1, 0, 0) in Fig. 1 are related to membership values of input fuzzy sets (A_1, A_0, A_0) of three rules in Table 1. The red component R of rule R^1 is fixed as fuzzy set A_1 (*not black*), and green component G of rule R^2 and blue component B of rule R^3 are fixed as fuzzy set A_0 (*white*) in Figs. 5a_2 and 6*b*.

The fuzzy inference process is as follows. Let the inputs be $c = c'$, $w = w'$, and $s = s'$. $U = (c', w', s')$.

1) In the input (antecedent) of rule R^k, the grade $\alpha_k = A_j(U')$, where k and j are shown in Table 1.

2) In the output (consequent) of rule R^k, the α_k level-set is shown as a vertical arrow.

3) $O_k' = \alpha_k O_k$, where O_k' are fuzzy sets and O_k are crisp sets (vertical poles) in Fig. 5*b*. The complete inference results O' of rules R^1, R^2, and R^3 are as follows:

$$O' = \alpha_1 O_1 \cup \alpha_2 O_2 \cup \alpha_3 O_3 = O_1' \cup O_2' \cup O_3' \qquad (8)$$

The output coordinate $V' = (r', g', b')$ is equivalent to the center of gravity of the output fuzzy set of O'. In addition, in Fig. 5*c*, $V' = (r', uk_o')$ corresponds to a coordinate of the graphic system, where uk' (on the unknown axis) is calculated from g' and b'. Furthermore, uk_o' describes a value (as distance from B) on the line B-G.

TABLE II.
MEMBERSHIP VALUE OF INPUT FUZZY SET A_j ON THE TONE TRIANGLE

Color	Color coordinate			Membership value μ_j			
	c	w	s	μ_1	$\mu_{0.5}$	$\mu_{0.25}$	μ_0
C	100	0	0	1.00	0.50	0.25	0.00
W	0	100	0	1.00	1.00	1.00	1.00
S	0	0	100	0.00	0.00	0.00	0.00

C is maximum chromaticness of each hue. S is black. Coordinates are indicated in %.

D. Fuzzy Sets

Table 2 shows the membership value $\mu_j(c', w', s')$ of the input fuzzy set A_j on the tone triangle. Here, $\mu_j(c', w', s')$ is equal to $\mu_j(c', uk_3')$, and uk_3 is equal to the lightness l. The membership function μ_j based on the *additive color mixing* was considered in the present study (see Fig. 6). Here, μ_j corresponds to A_j. Color coordinates c, w, and s in Table 2 are equivalent to No.66, No.11, and No.1, respectively, of the sixty-six crisp color inputs in each hue (e.g., red in Appendix D (Table 2) and Appendix E (Table 3)).

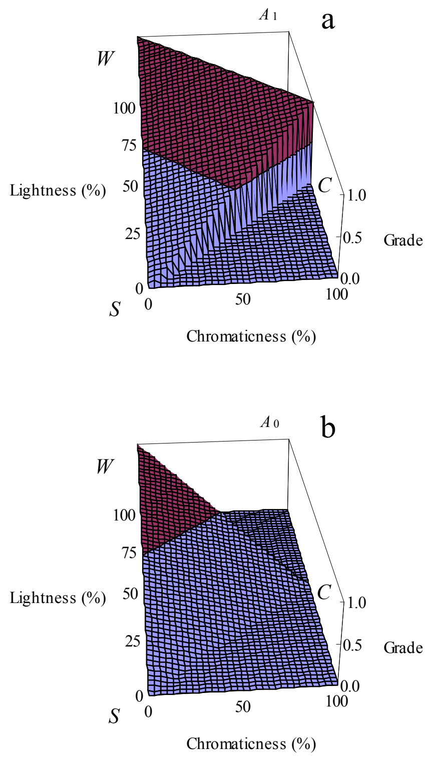


Figure 6. Membership functions *a*: $\mu_1(c, uk_3)$ of input fuzzy set A_1 (*not black*) and *b*: $\mu_0(c, uk_3)$ of input fuzzy set A_0 (*white*) on the tone triangle. uk_3 is equal to the lightness. See Appendix A (Fig. 1*a* and *b*).

As shown in Table 2, the membership value at corner C on the tone triangle forms the shape of the fuzzy set. The membership value (μ_1 - μ_0) at corner C is variable (1 – 0), that at W is fixed to 1, and that at S is fixed to 0 based on additive color mixing. For instance, in Table 1, if the hue is red, then the components are (R, G, B) = (1, 0, 0). These values correspond to the input fuzzy sets (A_1, A_0, A_0) of rules (R^1, R^2, R^3). Namely, A_1 has a membership value of 1 at corner C, and A_0 has a membership value of 0. In this case, A_0 is used twice in the rules R^2 and R^3.

Figure 6 shows lightness instead of whiteness and blackness in Fig. 5*a*, that is Fig. 6, is a general coordinate. The chromaticness is the same axis in Figs. 5*a* and 6. Figure 6 shows input fuzzy sets A_1 (*not black*) and A_0 (*white*). With the increasing blackness the membership value μ_j (*black* component) of input fuzzy set decreases in *a*. This forms an input fuzzy set of *not black*. On the other hand, with increasing whiteness the membership value μ_j (*white* component) of input fuzzy set increases in *b*. This forms an input fuzzy set of *white*. See also Table 1 and Figs. 5 and 8 (Appendix A (Fig. 1*a* and *b*)). It is interesting that the direction of chromaticness (or lightness) for the increasing membership value is not appropriate as input fuzzy set in this case.

In Table 1 (Appendix F (Table 4)), red has the components (R, G, B) = (1, 0, 0), which correspond to input fuzzy sets (A_1, A_0, A_0) of rules (R^1, R^2, R^3). In A_1 of Fig. 6*a* (see μ_1 in Table 2), if the membership value μ_1 at corner C has a membership value of 1, then the μ_1 at W has a membership value of 1 and μ_1 at S has a membership value of 0. Namely, in the input fuzzy set A_1, the components are (C, W, S) = (1, 1, 0). In the input fuzzy set A_0, the components are (C, W, S) = (0, 1, 0). Since the membership values in fuzzy sets (A_1, A_0, A_0) are (1, 1, 1) at corner W, they must be (0, 0, 0) at corner S. In other words, the red component R is composed of A_1 (*not black*), the green component G and the blue component B are composed of A_0 (*white*). The maximum chromaticness C corresponds to component R, G, or B. See Table 2.

E. Arrangements of Fuzzy Inputs

Figure 7*a* illustrates 21 fuzzy inputs (I_{46} through I_{66}) on parts of the tone triangle designated as dark, light, and C. The two modifiers are roughly fixed in their positions. The fuzzy inputs are formed by conical membership functions, and the fuzzy sets are made to mutually overlap. The edge of the basal plane (circle) of the conical fuzzy sets passes through the centers of the overlapped circles. It is easy to determine the changes to the inputs that are regularly arranged.

Figure 7*b* shows the arrangement of numbers corresponding to the conical fuzzy sets of Fig. 7*a*, and the numbers inside the circles represent the top of the 0.5 level-set (bottom-right). The color names and modifiers are No.46: dark (or deep), No.51: light (or pale), and No.66: maximum chromaticness C.

F. Membership Functions

Figure 8 shows half of the tone triangle as the base of input fuzzy set A_j and one of the sixty-six conical fuzzy inputs ($I_1 - I_{66}$) on the tone triangle. In the input fuzzy set A_1 (*not black*), the slope line shows a projection of the line between S with membership value $\mu_1 = 0$ and C with $\mu_1 = 1$ (or between S with membership value $\mu_1 = 0$ and W with $\mu_1 = 1$ on the other side), and in the input fuzzy set A_0 (*white*), the slope line shows a projection of the line between W with value $\mu_0 = 1$ and C with value $\mu_0 = 0$ (or between W with value $\mu_0 = 1$ and S with $\mu_0 = 0$ on the other side). See also Table 2 and Fig. 6. The triangular membership function Proj (I_{54}) on the blackness axis (*a*) and Proj (I_{54}) on the whiteness axis (*b*) is one of eleven projections of the sixty-six fuzzy inputs (I_1 through I_{66}) by the rays from the lower part, and the triangular membership function Proj (I_n) on the unknown axis (uk_1 and uk_2) is not used in the present study.

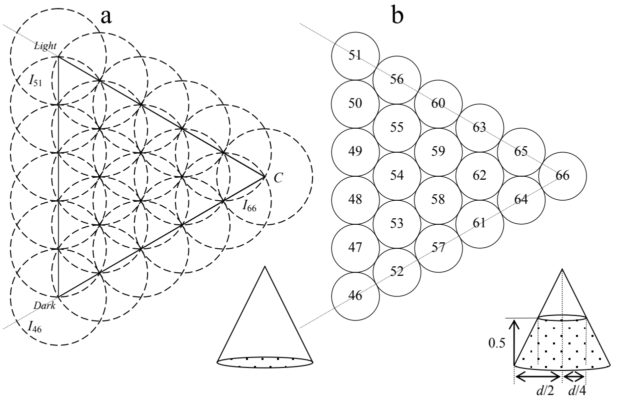


Figure 7. (*a*) Fuzzy inputs on part of the tone triangle and (*b*) top areas of 0.5 level-sets indicated by number. The diameter (d = 23.0%) of the basal plane (*circle*) of the cone indicates vagueness.

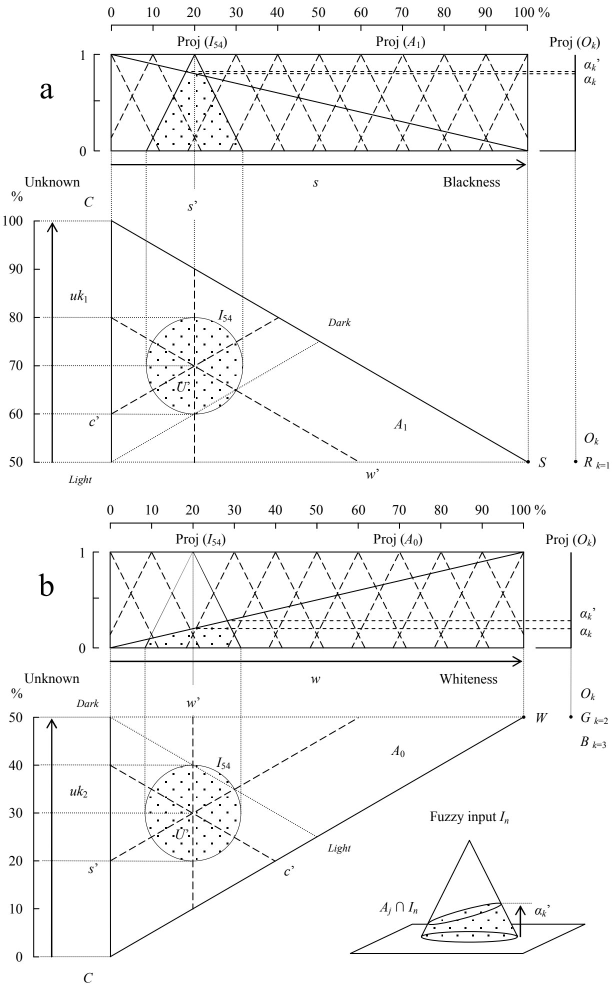


Figure 8. One of 66 conical fuzzy inputs (*vague colors*) and membership functions of input fuzzy sets A_j on half of the tone triangle. *a*: μ_1 (s, uk_1) = − 0.01s + 1 on the projection of A_1. *b*: μ_0 (w, uk_2) = 0.01w on the projection of A_0. See Appendix Fig. 1*a* and *b*.

An input fuzzy set A_1 of *not black* can be characterized by the following membership function:

$$\mu_1(s, uk_1) = -0.01s + 1 \quad (9)$$

where 0.01 is the slope of the projection. See Fig. 8*a*. The limitations of uk_1 (on the *C*-*W* side) are as follows:

$$50 \geq uk_1 \geq \frac{s}{2} \quad (10)$$

$$50 < uk_1 \leq -\frac{s}{2} + 100 \quad (11)$$

An input fuzzy set A_0 of *white* can be characterized by the following membership function:

$$\mu_0(w, uk_2) = 0.01w \quad (12)$$

where 0.01 is the slope of the projection. See Fig. 8*b*. The limitations of uk_2 (on the *S*-*C* side) are as follows:

$$50 \geq uk_2 \geq \frac{w}{2} \quad (13)$$

$$50 < uk_2 \leq -\frac{w}{2} + 100 \quad (14)$$

An input fuzzy set $A_{0.5}$ of *light* can be characterized by the following membership function:

$$\mu_{0.5}(c, uk_3) = 0.01uk_3 \quad (15)$$

where 0.01 is the slope of the projection on uk_3. The limitations of uk_3 (on the *W*-*S* side) are as follows:

$$50 \geq uk_3 \geq \frac{c}{2} \quad (16)$$

$$50 < uk_3 \leq -\frac{c}{2} + 100 \quad (17)$$

An input fuzzy set $A_{0.25}$ of 15-*degrees-light* can be characterized by the following membership function:

$$\mu_{0.25}(uk_5, uk_4) = 0.01uk_4 \quad (18)$$

where uk_5 and uk_4 rotate 15 degrees for c and uk_3, respectively, and 0.01 is the slope of the projection on uk_4. The limitations of uk_3 (on the *W*-*S* side) are as follows:

$$50 \geq uk_3 \geq \frac{c}{2} \quad (19)$$

$$50 < uk_3 \leq -\frac{c}{2} + 100 \quad (20)$$

$$c \geq 0 \quad (21)$$

When the output crisp set is at $R_{k=1}$ in Fig. 8*a* and is at $G_{k=2}$ and at $B_{k=3}$ in Fig. 8*b*, the hue is red.

For vague color, inputs to the tone triangle (Fig. 5*a*), the system outputs crisp colors on the RGB triangle (Fig. 5*b*) and on the graphic plane (Fig. 5*c*).

III. RESULTS AND DISCUSSION

What happens if a vague color is input into the tone triangular system? The system considered in the present study can translate input data *U* of a vague color to output data *V* of a simple color on the RGB triangle.

The intersection of input fuzzy set A_j for fuzzy input I_n is $A_j \cap I_n$. (See the dotted area at the bottom-right of Fig. 8*b*.) Grade α_k' = height ($A_j \cap I_n$). If the input is crisp, then α_k' becomes α_k. In Fig. 8*a*, *R* is the red output, and, in Fig. 8*b*, *G* and *B* are the green and blue outputs, respectively. Proj (O_k) is a projection of an output crisp set at corner *R*, *G*, or *B* (See Fig. 5*b*).

The fuzzy input (No.54) on the tone triangle is made up of the center *U*' = (*c*', *w*', *s*') = (60, 20, 20) in %, and the diameter *d* = 23.0% of the basal plane (circle) of the cone indicated vagueness.

In a previous study [4], the intersection of input fuzzy set A_j for fuzzy input I_n differed depending on whether I_n included the linear *edge* of A_j. The edges affected the nonlinear information processing. However, edge effects are not considered in the present study.

The inference results are shown only on the three center lines of the RGB triangle in Fig. 9. As Table 1 indicates, the three components are composed of two input fuzzy sets. Since two components are obtained by one fuzzy set (e.g., A_0), as the output fuzzy sets of the rules, the two vertical arrows in Fig. 5*b* have the same height because the intersections, as weights, are the same in Figs. 5*a* and 8.

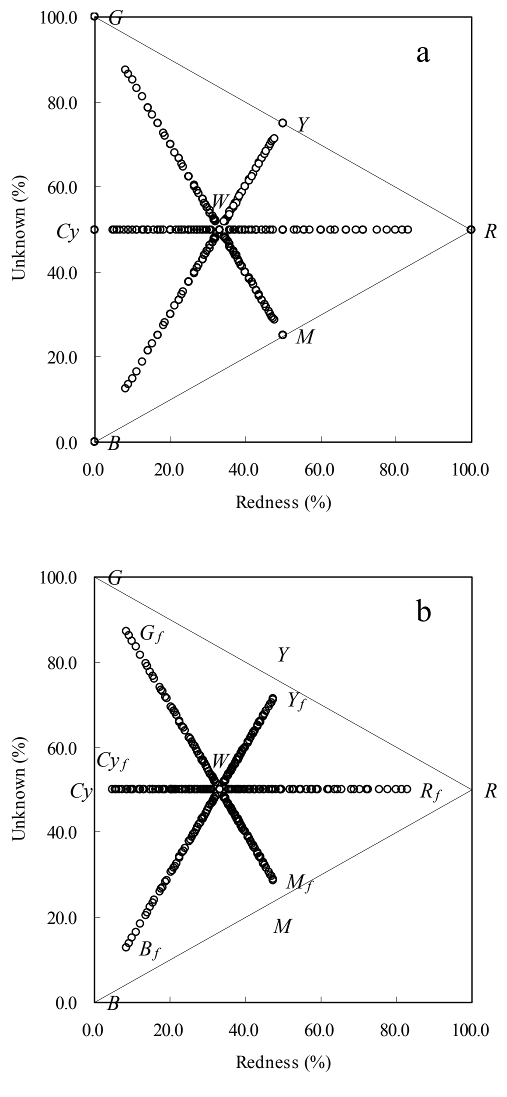


Figure 9. Inference outputs for (*a*) crisp inputs and (*b*) fuzzy inputs on the graphic plane. Suffix *f* indicates fuzzy inference output. White exists at the coordinates (33.3%, 50.0%).

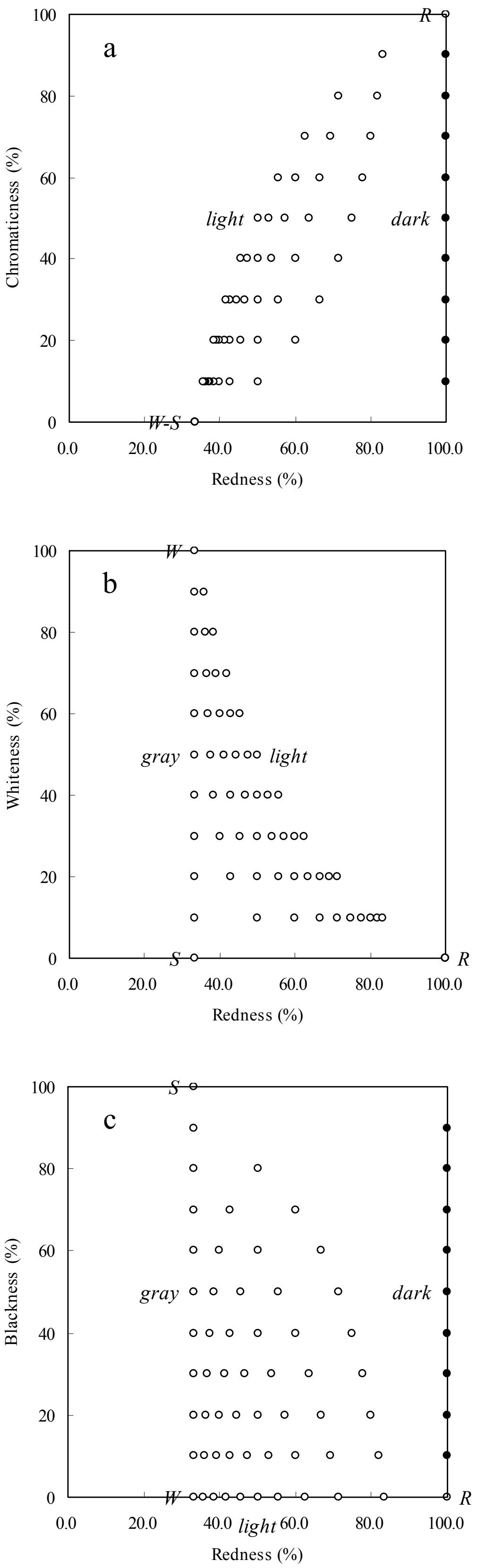


Figure 10. Relationship between (*a*) chromaticness, (*b*) whiteness, and (*c*) blackness as crisp inputs and redness as the inference outputs. *W-S* shows achromatic colors.

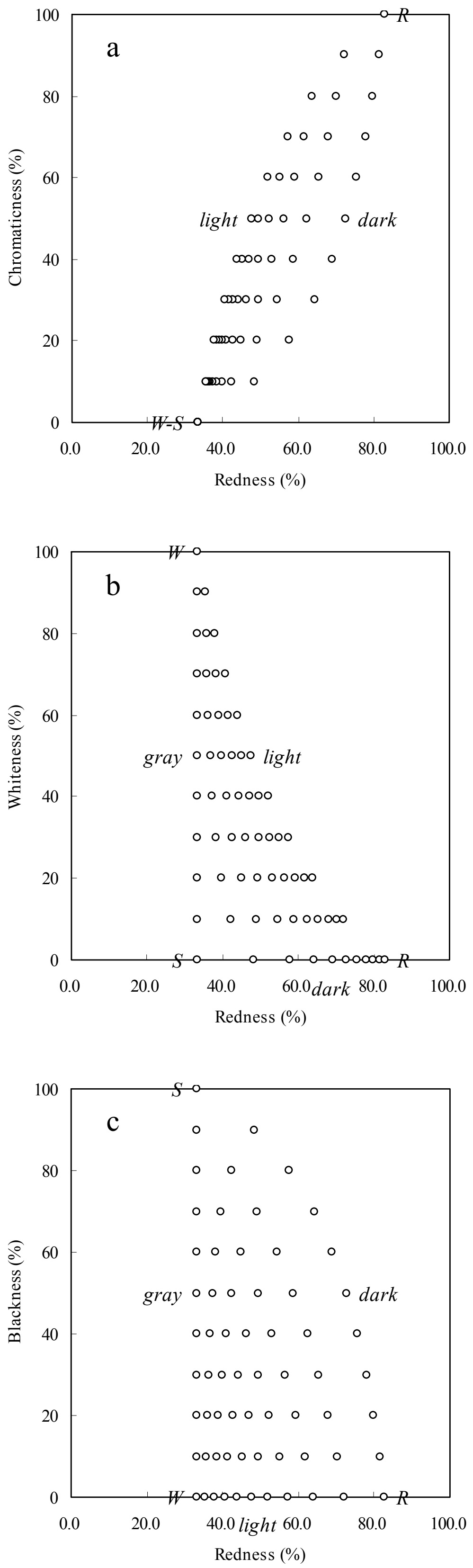


Figure 11. Relationship between (*a*) chromaticness, (*b*) whiteness, and (*c*) blackness as fuzzy inputs and redness as the inference outputs. *W-S* shows achromatic colors.

Figure 9*a* illustrates the relationship between the unknown value uk_o and the redness value r_o obtained from data (r_o', uk_o'). The circles indicate the outputs for crisp inputs of colors in Fig. 9*a*, corresponding to Fig. 5*c*. The inference outputs for crisp inputs are grouped at the center of the RGB triangle. This effect is caused by the shapes of the fuzzy set (triangular pyramid) and the computation of the center of gravity. These results are the same as the results of a previous study [4], [5], although the shapes of the triangular pyramids as the fuzzy sets are different, with the exception of the fuzzy sets of white. Namely, the gathering effects for crisp inputs using input fuzzy sets of a triangular pyramid were reported in the previous study [4], [5]. See Appendix D (Table 2).

Figure 9*b* also illustrates the relationship between the unknown value uk_o and the redness value r_o obtained from data (r_o', uk_o'). The circles indicate the outputs for fuzzy inputs of colors, corresponding to Fig. 5*c*. The inference outputs for fuzzy inputs are also gathered at the center of the RGB triangle. The inference outputs for crisp inputs in a are different from those for fuzzy inputs in b. However, the input-output (three attributes-redness) relationship cannot be seen in Fig. 9. See Appendix E (Table 3).

Figures 10 and 11 illustrate the relationship among the chromaticness, the whiteness, the blackness as the inputs, and the redness as the output. The outputs for crisp inputs are shown in Fig. 10 (as the input-output relationship). The outputs for fuzzy inputs are shown in Fig. 11. Figures 10*a* and 11*a* show the relationship between chromaticness and redness (leaf shape). With increasing chromaticness, the redness increases 100% (maximum) for crisp inputs (Fig. 10*a*) and 82.9% for fuzzy inputs (Fig. 11*a*). In contrast, with decreasing chromaticness, the redness converges at 33.3% (*W*-*S*) as the achromatic colors. Figures 10*b* and 11*b* show the relationship between whiteness and redness. The outputs for nine colors (Nos. 12, 22, 31, 39, 46, 52, 57, 61, and 64) and dark (No. 46) as a landmark modifier in Fig. 11*b* do not appear in between *S* and *R* in Fig. 10*b*. Figures 10*c* and 11*c* show the relationship between blackness and redness. With increasing blackness (or whiteness in *b*) the redness converges to 33.3% on the white (or black). In Fig. 11, the envelope curves (*R-light-W-S* in *a*, *W-light-R* in *b*, and *S-dark-R* in *c*) are similar. The outputs for whiteness and blackness show the same maximum values, i.e., 82.9%, but different characteristics, because the characteristics are caused by the shape of the fuzzy sets.

Redness outputs to fuzzy inputs show *natural* shapes (Fig. 11). In contrast, the outputs to crisp inputs, including to only the nine colors (filled circles), show *unnatural* shapes (Fig. 10). This is because the membership value $\mu_0 = 0$ from *S* to *C* in A_0 of Fig. 6*b*. The projections to the redness axis of Figs. 10 and 11 correspond to the redness axis of Fig. 9.

For example, the outputs of light and dark are fixed to the coordinates 50% and 100% for crisp inputs (Fig. 10) and 47.5% and 72.7% for fuzzy inputs (Fig. 11) on the redness axis. Since $\mu_1(c, uk_3)$ of input fuzzy set A_1 has a value of 1.0 (which is large) and $\mu_0(c, uk_3)$ of input fuzzy set A_0 has a value of 0.5 (which is moderate) at the general coordinate (50%, 75%) of light in Fig. 6, the inference output (as the center of gravity of intersections α_k calculated with three membership values μ_j) is 50% or 47.5% (which is small) on the redness axis. The output is far from the corners (or sides) of the triangle (Fig. 9*a*). See Appendix D (Table 2) and Appendix E (Table 3) No. 51 (light). On the other hand, since $\mu_1(c, uk_3)$ has a value of 0.5 (moderate) and $\mu_0(c, uk_3)$ has a value of zero at the coordinate (50%, 25%) of dark in Fig. 6, the inference output is 100% or 72.7% (large) on the redness axis. The output is at the corner *R* of the triangle (Fig. 9*a*). This calculation is based on only μ_1. See Appendix D (Table 2) and Appendix E (Table 3) No. 46 (dark).

Figure 12 shows input fuzzy sets $A_{0.5}$ (*light*) and $A_{0.25}$ (15-*degree-light*). With increasing lightness the membership value μ_j (*light* component) of the input fuzzy set increases in *a*. This forms an input fuzzy set of *light*. On the other hand, with increasing 15-degree-lightness the membership value μ_j (15-*degree-light* component) of the input fuzzy set increases in *b*. This forms an input fuzzy set of 15-*degree-light*. See Appendix A (Fig. 1*c* and *d*). It is interesting that the direction of chromaticness for the increasing membership value is not appropriate as a input fuzzy set in this case.

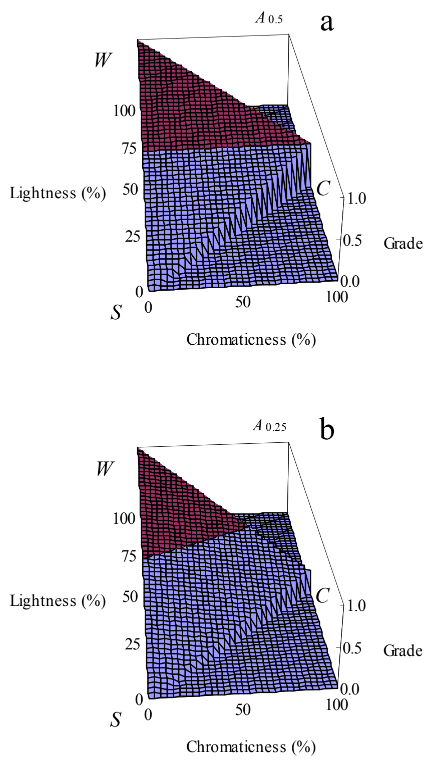


Figure 12. Membership functions *a*: $\mu_{0.5}(c, uk_3)$ of input fuzzy set $A_{0.5}$ (*light*) and *b*: $\mu_{0.25}(c, uk_3)$ of input fuzzy set $A_{0.25}$ (15-*degree-light*) on the tone triangle. uk_3 is equal to the lightness. See Appendix Fig. 1*c* and *d*.

Table 3 shows the fuzzy rules of three other colors. Three rules (R^1, R^2, R^3) are composed of two or three input fuzzy sets (Figs. 6 and 12). Note that the input fuzzy sets of rules in Table 3 are different from those in Table 1.

In Table 3 as the membership value on the maximum chromaticness C is dependent upon the components of hue, for example orange in Table 3 has the components (R, G, B) = (1, 0.5, 0), corresponding to input fuzzy sets (A_1, $A_{0.5}$, A_0) of the rules. Red component R is fixed as fuzzy set A_1 (*not black*). Green component G is fixed as fuzzy set $A_{0.5}$ (*light*). Finally, blue component B is fixed as fuzzy set A_0 (*white*). In addition, lime has the components (R, G, B) = (0.5, 1, 0). See Appendix C (Table 1).

TABLE III.
FUZZY RULES FOR THREE OTHER COLORS

Hue color	Input fuzzy set			Output crisp set		
	R^1	R^2	R^3	R^1	R^2	R^3
Orange	A_1	$A_{0.5}$	A_0	O_1	O_2	O_3
Lime	$A_{0.5}$	A_1	A_0	O_1	O_2	O_3
Brown	A_1	$A_{0.25}$	$A_{0.25}$	O_1	O_2	O_3

$A_{0.5}$ and $A_{0.25}$ are the shapes between A_1 and A_0. Membership values of $A_{0.5}$ and $A_{0.25}$ are 0.5 and 0.25 at coordinates (100, 0, 0). See Fig. 13.

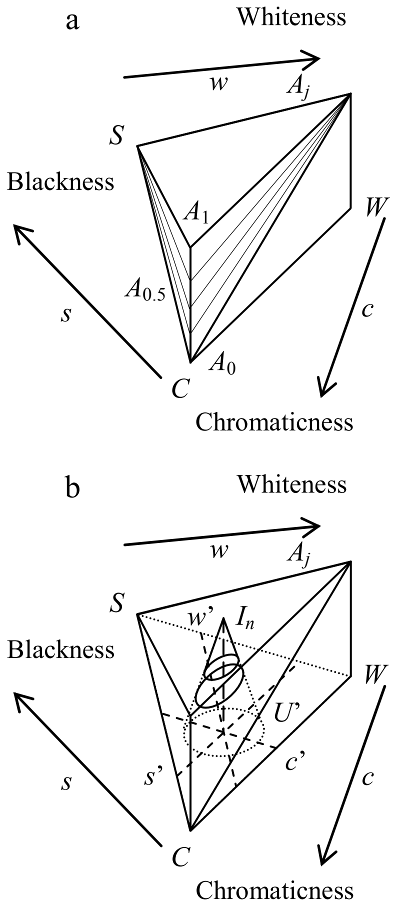


Figure 13. (*a*) Difference of input fuzzy sets A_1, $A_{0.75}$, $A_{0.5}$, $A_{0.25}$, and A_0 and (*b*) intersections of input fuzzy sets A_1 and A_0 to the pole crisp input or the conical fuzzy input I_n on the tone triangle.

Figure 13 shows the combination of fuzzy sets in Figs. 6 and 12 and the difference of the intersections of fuzzy sets in Figs. 6*a* and *b*. Fuzzy set A_1 (*not black*) is used as the main component for the six fundamental colors (Table 1), and the membership value μ_1 on the maximum chromaticness C is equal to 1. Fuzzy set A_0 (*white*) is used as the sub components, and the membership value μ_0 on the maximum chromaticness C is equal to 0 (Table 2). In addition, $A_{0.5}$ (*light*) and $A_{0.25}$ (15-*degree-light*) are 0.5 and 0.25 (Table 2), which are used as the sub components for three other colors (Table 3). For five overlapping input fuzzy sets (A_1, $A_{0.75}$, $A_{0.5}$, $A_{0.25}$, A_0), the shapes of the input fuzzy sets are given (fixed) by the size of the membership value on the maximum chromaticness C. As shown in Fig. 13*b*, the intersections of conical fuzzy inputs to the input fuzzy sets of the triangular pyramid can be easily computed. The difference of the intersections is large at corner C. If the chromaticness is increasing, i.e., three weights α_k (k = 1, 2, 3) on the RGB triangle are unbalanced, as shown in Fig. 5*b*, then the inference results are close to the sides (or corners) of the triangle, which, with the exception of brown, is not far from the sides (or corners) (Fig. 9*a*).

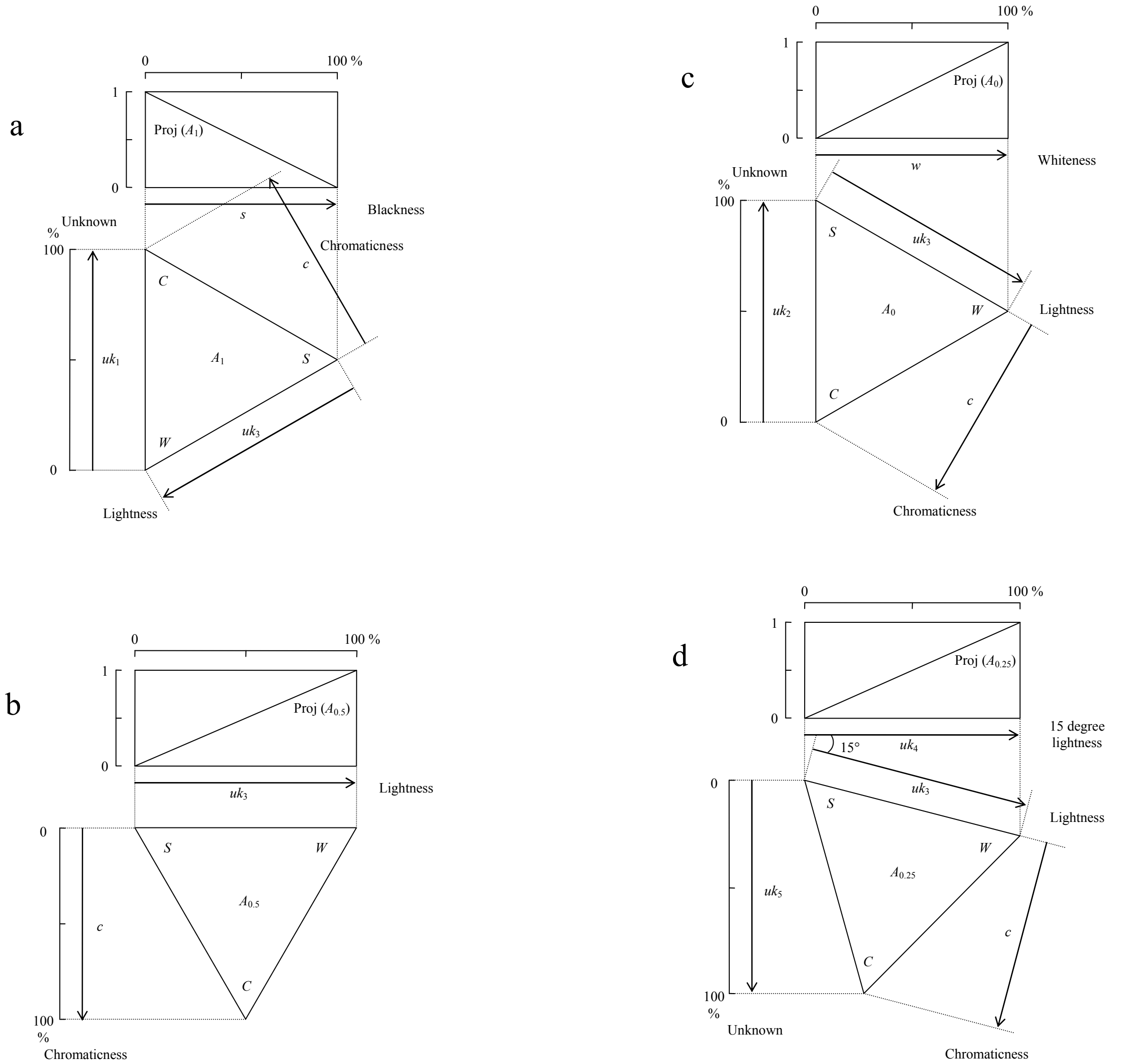
Color is well defined [13]. That is, color is not vague or fuzzy. However color words (e.g., red) are vague or fuzzy. Human responses for color identification are vague or fuzzy, for example, red-relevant colors with different grades show fuzzy set like a mountain shape [10]. Fuzzy sets provide a mathematical way to represent vagueness and fuzziness in humanistic system [2]. This fuzzy system is useful to evaluate human responses as vague color information processing. Stocking such a semantic data the human-computer interaction will be going well with the environment.

IV. Conclusions

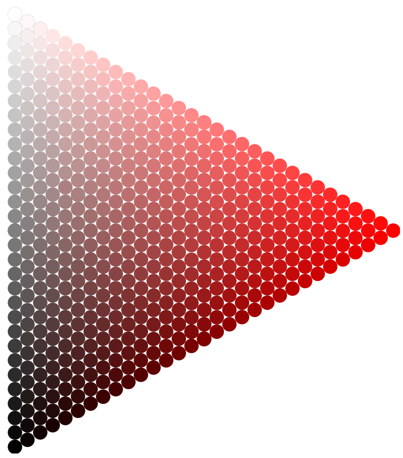
The present paper proposes a fuzzy system that can extract crisp outputs on the RGB triangle (which is available for use in fuzzy set theory) and a graphic system (which is easy to show via graphs). The proposed system also extracts, in a simple manner, the membership values from the projection of a conical membership function of a vague color input. Three weight parameters associated with respective grades indicate vague colors and output the center of gravity as a crisp color value. Namely, the system can map the tone information of a color space (3D cubic) to an RGB triangle (2D planar). However, these parameters are gathering points, and the detailed behaviors are not apparent. Therefore, the relationship among chromaticness, whiteness, and blackness as the inputs and redness as the output is clarified in this case [9]. In particular, the outputs to fuzzy inputs show characteristics of *natural* shape, and the outputs to crisp inputs show characteristics of *unnatural* shape, especially, at *C*-*S* area (including to dark).

The values of membership functions at corner C are completely proportional to the components of RGB. All of the colors can be constructed by the fuzzy rule. In addition, the membership values (as human subjectivity) on the tone triangle [10] or on the color triangle [8], [11] are analyzed using fuzzy inference.

APPENDIX A FIGURE 1. TOP VIEWS AND PROJECTIONS OF INPUT FUZZY SETS A_1 (A: *NOT BLACK*), A_0 (B: *WHITE*), $A_{0.5}$ (C: *LIGHT*), AND $A_{0.25}$ (D: 15-*DEGREE-LIGHT*) ON THE TONE TRIANGLE WITH ATTRIBUTES.



APPENDIX B FIGURE 2. CRISP COLOR INPUTS (496, DETAIL TYPE) ON THE TONE TRIANGLE. THE DETAIL TYPE (496 COLORS) CORRESPONDS TO THE FUNDAMENTAL TYPE (66 COLORS).



APPENDIX C TABLE 1 COMPONENTS AND COORDINATES OF HUE COLORS AND WHITE

Color	Component			Coordinate		
	R	*G*	*B*	*r*	*g*	*b*
Red	1	0	0	1	0	0
Green	0	1	0	0	1	0
Blue	0	0	1	0	0	1
Yellow	1	1	0	1/2	1/2	0
Cyan	0	1	1	0	1/2	1/2
Magenta	1	0	1	1/2	0	1/2
Orange	1	1/2	0	2/3	1/3	0
Lime	1/2	1	0	1/3	2/3	0
Brown	1	1/4	1/4	2/3	1/6	1/6
White	1	1	1	1/3	1/3	1/3

By substituting the components into Eqs. (2) through (4), the coordinates are computed and Eq. (5) is satisfied.

APPENDIX D TABLE 2 INFERENCE RESULTS FOR CRISP INPUTS OF TONE COLORS IN RED

No.	Crisp input			Grade for crisp input			Inference output			Graphic output	
	c’	*w*’	*s*’	α_1	α_2	α_3	r_o’	g_o’	b_o’	r_o’	uk_o’
1	0	0	100	0.00	0.00	0.00	33.3	33.3	33.3	33.3	50.0
2	0	10	90	0.10	0.10	0.10	33.3	33.3	33.3	33.3	50.0
3	0	20	80	0.20	0.20	0.20	33.3	33.3	33.3	33.3	50.0
4	0	30	70	0.30	0.30	0.30	33.3	33.3	33.3	33.3	50.0
5	0	40	60	0.40	0.40	0.40	33.3	33.3	33.3	33.3	50.0
6	0	50	50	0.50	0.50	0.50	33.3	33.3	33.3	33.3	50.0
7	0	60	40	0.60	0.60	0.60	33.3	33.3	33.3	33.3	50.0
8	0	70	30	0.70	0.70	0.70	33.3	33.3	33.3	33.3	50.0
9	0	80	20	0.80	0.80	0.80	33.3	33.3	33.3	33.3	50.0
10	0	90	10	0.90	0.90	0.90	33.3	33.3	33.3	33.3	50.0
11	0	100	0	1.00	1.00	1.00	33.3	33.3	33.3	33.3	50.0
-	-	-	-	-	-	-	-	-	-	-	-
46	50	0	50	0.50	0.00	0.00	100.0	0.0	0.0	100.0	50.0
47	50	10	40	0.60	0.10	0.10	75.0	12.5	12.5	75.0	50.0
48	50	20	30	0.70	0.20	0.20	63.6	18.2	18.2	63.6	50.0
49	50	30	20	0.80	0.30	0.30	57.1	21.4	21.4	57.1	50.0
50	50	40	10	0.90	0.40	0.40	52.9	23.5	23.5	52.9	50.0
51	50	50	0	1.00	0.50	0.50	50.0	25.0	25.0	50.0	50.0
-	-	-	-	-	-	-	-	-	-	-	-
66	100	0	0	1.00	0.00	0.00	100.0	0.0	0.0	100.0	50.0

APPENDIX E TABLE 3 INFERENCE RESULTS FOR FUZZY INPUTS OF TONE COLORS IN RED

No.	Center of fuzzy input			Grade for fuzzy input			Inference output			Graphic output	
	c’	*w*’	*s*’	α_1’	α_2’	α_3’	r_o’	g_o’	b_o’	r_o’	uk_o’
1	0	0	100	0.10	0.10	0.10	33.3	33.3	33.3	33.3	50.0
2	0	10	90	0.19	0.19	0.19	33.3	33.3	33.3	33.3	50.0
3	0	20	80	0.28	0.28	0.28	33.3	33.3	33.3	33.3	50.0
4	0	30	70	0.37	0.37	0.37	33.3	33.3	33.3	33.3	50.0
5	0	40	60	0.46	0.46	0.46	33.3	33.3	33.3	33.3	50.0
6	0	50	50	0.55	0.55	0.55	33.3	33.3	33.3	33.3	50.0
7	0	60	40	0.64	0.64	0.64	33.3	33.3	33.3	33.3	50.0
8	0	70	30	0.73	0.73	0.73	33.3	33.3	33.3	33.3	50.0
9	0	80	20	0.82	0.82	0.82	33.3	33.3	33.3	33.3	50.0
10	0	90	10	0.91	0.91	0.91	33.3	33.3	33.3	33.3	50.0
11	0	100	0	1.00	1.00	1.00	33.3	33.3	33.3	33.3	50.0
-	-	-	-	-	-	-	-	-	-	-	-
46	50	0	50	0.55	0.10	0.10	72.7	13.6	13.6	72.7	50.0
47	50	10	40	0.64	0.19	0.19	62.4	18.8	18.8	62.4	50.0
48	50	20	30	0.73	0.28	0.28	56.4	21.8	21.8	56.4	50.0
49	50	30	20	0.82	0.37	0.37	52.4	23.8	23.8	52.4	50.0
50	50	40	10	0.91	0.46	0.46	49.6	25.2	25.2	49.6	50.0
51	50	50	0	1.00	0.55	0.55	47.5	26.2	26.2	47.5	50.0
-	-	-	-	-	-	-	-	-	-	-	-
66	100	0	0	1.00	0.10	0.10	82.9	8.6	8.6	82.9	50.0

In Appendixes D and E (Tables 2 and 3), the color names or modifiers are No.1: black, No.6: gray, No.11: white, No.46: dark (or deep), No.51: light (or pale), and No.66: *C*, maximum chromaticness [3] (e.g., vivid red). No.12 through No.45 and No.52 through No.65 are not shown in this case. Although the intersections α_k of zero based on the shapes of the input fuzzy sets cannot compute the results at black (No.1) in Table 2, the inference output (r_o', g_o', b_o') should be fixed (33.3, 33.3, 33.3) in % because achromatic colors (No.1 through No.11), including black, must be positioned at the center of the triangle (See Fig. 3*b*).

APPENDIX F TABLE 4 AN EXAMPLE OF FUZZY SETS IN HUE COLOR (RED)

$(R^1,\_R^2,\_R^3)$	$(A_1,$	$A_0,$	$A_0)$	Hue color
(*C*, *W*, *S*)	(1, 1, 0)	(0, 1, 0)	(0, 1, 0)	
C	1	0	0	Red
W	1	1	1	White
S	0	0	0	Black
	R	*G*	*B*	

The RGB components of white and black in shaded area are fixed.

ACKNOWLEDGMENT

The author wishes to thank the current and former members of our laboratory, whose work and ideas have contributed to this project.

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